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THE NONLOCAL INFLUENCE OF DENSITY VARIATIONS IN A COMPOSITE

J. R. Willis
School of Mathematics, University of Bath, Claverton Down, Bath BA2 7AY, U.K.

Abstract—In any randomly inhomogeneous elastic medium, it is well known that the mean stress may be related to the mean strain through a nonlocal operator, which reduces to an ordinary tensor of moduli when applied to a slowly-varying mean field. This study discusses a corresponding relation between the mean momentum density and the mean velocity. Attention is focused on this feature of overall behaviour by considering a model medium in which only the density varies. When the medium is statistically uniform, the "overall mass density" operator has the form of a convolution in time and space and it is convenient to discuss its combined Laplace and Fourier transform. The operator is introduced via a brief consideration of its representation as a perturbation series but the bulk of the work is devoted to establishing bounds of "Hashin-Shtrikman" type for the eigenvalues of its combined transform, when the transform variables are real. Explicit bounds for a two-phase medium with a particular form for its two-point correlation function are given. They define the eigenvalues within 0.5 percent when the ratio of the two densities is two and remain close enough to give a good indication of overall behaviour at density ratios as high as ten.

1. INTRODUCTION

The motion of a body is governed by the equation of motion

$$\operatorname{div} \sigma + f = \dot{p}, \tag{1.1}$$

where σ denotes the stress tensor, f represents body force, p is momentum density, and the superposed dot represents differentiation with respect to time. If the body is elastic, with tensor of elastic moduli L and mass density ρ, then σ and p are related to the displacement u through the constitutive relations

$$\sigma = Le, \qquad p = \rho\dot{u}, \tag{1.2}$$

where e represents the infinitesimal strain tensor associated with u. The motion is completely specified by (1.1), (1.2), plus suitable initial and boundary conditions.

Suppose now that the body is a composite, so that L and ρ vary in a complicated manner with position x. It is common to view the body as a random medium and, instead of solving (1.1), (1.2), to attempt to construct equations which determine the ensemble average $\langle u\rangle$ of u. Equation (1.1) can be averaged immediately to give

$$\operatorname{div} \langle\sigma\rangle + f = \langle\dot{p}\rangle. \tag{1.3}$$

Ensemble averaging (1.2) gives relations of the same form, namely

$$\langle\sigma\rangle = \hat{L}\langle e\rangle, \qquad \langle p\rangle = \hat{\rho}\langle\dot{u}\rangle, \tag{1.4}$$

if $\hat{L}$, $\hat{\rho}$ are *defined* so that

$$\hat{L}\langle e\rangle = \langle Le\rangle, \qquad \hat{\rho}\langle\dot{u}\rangle = \langle\rho\dot{u}\rangle. \tag{1.5}$$

The entities $\hat{L}$, $\hat{\rho}$ defined by (1.5) can be termed "overall" properties of the composite.

In the case of elastostatics, it is well known that (1.5) defines $\hat{L}$ as a non-local operator (see, for example, Beran and McCoy[1], Diener, Hürrich and Weissbarth[2], Willis[3]), which reduces to an ordinary tensor of moduli when applied to any field $\langle e\rangle$

that varies sufficiently slowly relative to the microscale of the composite. For dynamic problems, however, the usual approach is to seek directly equations which define the mean wave $\langle u \rangle$ without assigning a meaning to individual terms (see, for example, Karal and Keller[4], McCoy[5], Varadan, Varadan and Pao[6], Devaney[7], Tsang and Kong[8]). The representation of $\hat{\rho}$ is discussed in this article. In fact, attention is focused exclusively on $\hat{\rho}$, by considering an artificial "model" medium, in which the mass density ρ varies but the modulus tensor L does not. Thus, averaging the first of equations (1.2) gives

$$\langle \sigma \rangle = L\langle e \rangle, \tag{1.6}$$

so that $\hat{L}$ reduces to the ordinary tensor L or, if it is still considered as an operator, L multiplied by a delta function.

Section 2 discusses the representation of $\hat{\rho}$ from the standpoint of perturbation theory, valid for small fluctuations in ρ. Then, in Sections 3 and 4, its characterization by use of the variational principle of Willis[9] is considered. Bounds are obtained for a quadratic form associated with the Laplace transform of $\hat{\rho}$ (taken with respect to time) and these induce bounds on corresponding Fourier components. The procedure is analogous to one employed to characterize $\hat{L}$ in the case of elastostatics by Diener, Hürrich and Weissbarth[2] but the reasoning as presented is more closely related to that of Willis[3]. Section 5 discusses the particular case of a two-component material, for which results can be expressed in terms of a single two-point correlation function. Calculations performed for a particular choice of this function yield bounds that differ by less than one percent when the ratio of the two densities is two, by less than five percent at a density ratio of four, and by up to twenty-five percent at a density ratio of ten. Thus, for a practically useful range of density ratios, the Laplace transform of $\hat{\rho}$ is characterized quite accurately, independently of higher-order statistics, but knowledge of the latter would be required to achieve corresponding accuracy at very high density ratios.

2. PERTURBATION THEORY

This section essentially employs the method of smoothing of Karal and Keller[4] but it is convenient to present this in a manner that introduces notation used later. An infinite body is considered, in which the displacement u is generated by a body force f which decays suitably at large $|x|$. Combining (1.1) and (1.2) gives the usual equation of motion

$$\operatorname{div}(Le) + f = \rho \ddot{u}, \tag{2.1}$$

which can be written in the alternative form

$$\operatorname{div}(Le) + f - \dot{\pi} = \rho_0 \ddot{u}, \tag{2.2}$$

where ρ_0 is the density of a "comparison medium," which will be taken uniform, and the "momentum polarization" π, which was introduced by Willis[10, 11], is defined as

$$\pi = (\rho - \rho_0)\dot{u}. \tag{2.3}$$

In terms of the Green's function $G_0(x, t)$ of the comparison medium, the solution of (2.2) is

$$u(x, t) = \int \mathrm{d}x' \int \mathrm{d}t'\, G_0(x - x', t - t')\,[f(x', t') - \dot{\pi}(x', t')]$$

or, more briefly,

$$u = G_0(f - \dot{\pi}), \tag{2.4}$$

so that, in this context, G_0 is an operator. Then, from (2.3) and (2.4), π satisfies the equation

$$\pi + (\rho - \rho_0)\ddot{G}_0\pi = (\rho - \rho_0)\dot{G}_0 f. \tag{2.5}$$

It is convenient to eliminate f from (2.5) by subtracting from it the time derivative of the average of (2.4). Thus,

$$\pi + (\rho - \rho_0)\ddot{G}_0 (\pi - \langle\pi\rangle) = (\rho - \rho_0)\langle\dot{u}\rangle. \tag{2.6}$$

Equation (2.6) can be solved by iteration, if $\rho - \rho_0$ is small; truncating at the first non-trivial term gives

$$\pi = [\rho - \rho_0 - (\rho - \rho_0)\ddot{G}_0 (\rho - \langle\rho\rangle)]\langle\dot{u}\rangle. \tag{2.7}$$

Now from (1.5) and (2.3),

$$\hat{\rho}\langle\dot{u}\rangle = \rho_0\langle\dot{u}\rangle + \langle\pi\rangle. \tag{2.8}$$

Therefore, from (2.7) and (2.8),

$$\hat{\rho} = \langle\rho\rangle - \langle(\rho - \langle\rho\rangle)\ddot{G}_0 (\rho - \langle\rho\rangle)\rangle, \tag{2.9}$$

approximately, when ρ varies only slightly. In (2.9), symmetry has been maintained by replacing a factor ρ_0 by $\langle\rho\rangle$; the averaging ensures that this factor is actually multiplied by zero. This allows the right side of (2.9) to be expressed in terms of the correlation coefficient

$$h(x, x') = \frac{\langle(\rho(x) - \langle\rho(x)\rangle)(\rho(x') - \langle\rho(x')\rangle)\rangle}{\langle\rho^2(x)\rangle - \langle\rho(x)\rangle^2}. \tag{2.10}$$

If the medium is statistically uniform, $\langle\rho(x)\rangle$ and $\langle\rho^2(x)\rangle$ are independent of x, and h depends upon x, x' only in the combination $x - x'$. In this case, (2.9) can be given in the explicit form

$$\hat{\rho}\langle\dot{u}\rangle = \langle\rho\rangle\langle\dot{u}\rangle - (\langle\rho^2\rangle - \langle\rho\rangle^2)\,\ddot{H}\,\langle\dot{u}\rangle, \tag{2.11}$$

where H is a convolution operator, like G_0, with kernel

$$H(x, t) = G_0(x, t)h(x). \tag{2.12}$$

This result is equivalent to one given by Karal and Keller[4]. In its present form, it demonstrates explicitly that $\hat{\rho}$ is a nonlocal operator; also, it is expressed in notation that will be employed in the succeeding sections, which consider the representation of $\hat{\rho}$ when the density fluctuations may be large.

3. VARIATIONAL FORMULATION

The stationary principle

$$\delta\mathcal{H}(\pi) = 0, \tag{3.1}$$

where

$$\mathcal{H}(\pi) = \int dx\,[\pi * \dot{G}_0 f - \tfrac{1}{2}\pi * (\rho - \rho_0)^{-1}\pi - \tfrac{1}{2}\pi * \ddot{G}_0\pi], \tag{3.2}$$

the symbol $*$ denoting convolution with respect to time, follows directly from the integral equation (2.5). It is a special case of a principle given by Willis[9], who also showed that both a maximum and a minimum principle could be generated for the Laplace transform of $\mathcal{H}$, by suitable choices of ρ_0. The reasoning of Willis[9] can be summarized quite simply in the present case. From this point onwards, only Laplace transformed equations will be considered and, to avoid introducing new symbols, the Laplace transform of a function $g(x, t)$ will be written as $g(x, s)$ or, still more briefly when the context makes the meaning clear, simply as g. Also, the convolution $*$ becomes an ordinary product, so that the $*$ will be dispensed with.

When the transform variable s is real, the Laplace transform of the system (2.1) implies the minimum energy principle

$$\mathcal{F}(u) \leq \mathcal{F}(u^*), \tag{3.3}$$

where u is the actual solution, u^* is any field which decays suitably at large $|x|$ and

$$\mathcal{F}(u^*) = \int \mathrm{d}x \, [\tfrac{1}{2}e^*Le^* + \tfrac{1}{2}s^2\rho u^{*2} - fu^*], \tag{3.4}$$

e^* being the strain associated with u^*. Since u satisfies the Laplace transform of (2.1), it follows that

$$\mathcal{F}(u) = -\frac{1}{2}\int \mathrm{d}x \, fu. \tag{3.5}$$

Now, motivated by (2.4), take

$$u^* = G_0(f - s\pi^*) \tag{3.6}$$

and define

$$\sigma^* = Le^*, \qquad p^* = s\rho_0 u^* + \pi^*. \tag{3.7}$$

Then

$$\operatorname{div} \sigma^* + f = sp^* \tag{3.8}$$

for any choice of π^*, and π^* is the actual polarization π if it satisfies

$$\pi^* = s(\rho - \rho_0)u^*. \tag{3.9}$$

In this case, $u^* = u$, $\sigma^* = \sigma$, and $p^* = p$ also. Elementary manipulation, using Gauss' theorem with eqns (3.7), (3.8), yields

$$\mathcal{F}(u) = -\frac{1}{2}\int \mathrm{d}x \, fu \leq \mathcal{H}(\pi^*) - \frac{1}{2}\int \mathrm{d}x \, fG_0 f + \frac{1}{2}\int \mathrm{d}x \, [s(\rho - \rho_0)u^* - \pi^*]^2/(\rho - \rho_0), \tag{3.10}$$

where $\mathcal{H}(\pi^*)$ now represents the Laplace transform

$$\mathcal{H}(\pi^*) = \int \mathrm{d}x \left[s\pi^* G_0 f - \tfrac{1}{2}\pi^{*2}/(\rho - \rho_0) - \frac{s^2}{2}\pi^* G_0 \pi^* \right]. \tag{3.11}$$

Therefore, if ρ_0 is chosen so that $\rho_0 \geq \rho$ at each point x (with the agreement that π^*

$= 0$ wherever $\rho_0 = \rho$), it follows from eqn (3.10) that

$$\mathcal{F}(u) = -\frac{1}{2}\int dx\, fu \le \mathcal{H}(\pi^*) - \frac{1}{2}\int dx\, fG_0f. \tag{3.12}$$

Still keeping s real, the Laplace transform of the system (2.1) also implies the complementary energy principle

$$\mathcal{G}(\sigma, p) \le \mathcal{G}(\sigma^*, p^*), \tag{3.13}$$

where

$$\mathcal{G}(\sigma^*, p^*) = \frac{1}{2}\int dx\, (\sigma^* L^{-1}\sigma^* + p^{*2}/\rho), \tag{3.14}$$

for any pair of fields (σ^*, p^*) that satisfy

$$\operatorname{div} \sigma^* + f = sp^*. \tag{3.15}$$

If σ^*, p^* are generated from π^* via eqns (3.6), (3.7), manipulations similar to those described above give

$$\mathcal{G}(\sigma,p) = \frac{1}{2}\int dx\, fu \le \frac{1}{2}\int dx\, fG_0f - \mathcal{H}(\pi^*) - \frac{1}{2}\int dx \frac{\rho_0[s(\rho-\rho_0)u^* - \pi^*]^2}{\rho(\rho-\rho_0)}. \tag{3.16}$$

Therefore, if ρ_0 is chosen so that $\rho_0 \le \rho$ everywhere,

$$\mathcal{G}(\sigma, p) = \frac{1}{2}\int dx\, fu \le \frac{1}{2}\int dx\, fG_0f - \mathcal{H}(\pi^*). \tag{3.17}$$

Reasoning of this kind was first given by Hill[12] in the context of elastostatics, the analogue of (3.1) being the variational principle of Hashin and Shtrikman[13].

Since inequalities are not disturbed by ensemble averaging, it follows from eqns (3.12) and (3.17) that

$$\frac{1}{2}\int dx\, f\,\langle u\rangle \underset{(\le)}{\ge} \frac{1}{2}\int dx\, fG_0f - \langle\mathcal{H}(\pi^*)\rangle, \tag{3.18}$$

so long as ρ_0 is chosen so that $\rho_0 \ge (\le)\ \rho$ at each point x, for every possible realization of the composite. Here, π^* is any trial polarization, which may be chosen differently for each realization of the composite.

4. BOUNDS

Now specialize to a composite which consists of n constituents, or phases, the rth phase having density ρ_r, though all have the same tensor of moduli L. The density at x is conveniently represented in the form

$$\rho(x) = \sum_{r=1}^{n} \rho_r f_r(x), \tag{4.1}$$

where

$$f_r(x) = 1 \quad \text{if } x \in \text{phase } r$$
$$= 0 \quad \text{otherwise}. \tag{4.2}$$

Correspondingly, consider the field

$$\pi^* = \sum_{r=1}^{n} \pi_r(x) f_r(x), \tag{4.3}$$

where the $\pi_r(x)$ are functions of x only, so far unspecified. Now the probability of finding phase r at x is

$$p_r(x) = \langle f_r(x) \rangle \tag{4.4}$$

and the probability of finding phase r at x and phase s at x' is

$$p_{rs}(x, x') = \langle f_r(x) f_s(x') \rangle. \tag{4.5}$$

If the composite is statistically uniform, p_r is independent of x and p_{rs} depends only on $(x - x')$. When eqn (4.3) is substituted into $\langle \mathcal{H} \rangle$, there results

$$\langle \mathcal{H}(\pi^*) \rangle = \int \mathrm{d}x \left\{ \sum_{r=1}^{n} \left[s p_r \pi_r G_0 f - \tfrac{1}{2} p_r \pi_r^2 / (\rho_r - \rho_0) - \frac{s^2}{2} \pi_r \sum_{s=1}^{n} G_0 p_{rs} \pi_s \right] \right\}. \tag{4.6}$$

This quadratic expression involving the n functions $\pi_r(x)$ is stationary when the π_r satisfy

$$p_r \pi_r / (\rho_r - \rho_0) + s^2 \sum_{s=1}^{n} G_0 p_{rs} \pi_s = s p_r G_0 f; \tag{4.7}$$

the corresponding stationary value is

$$\frac{s}{2} \int \mathrm{d}x \, \langle \pi^* \rangle \, G_0 f = \frac{s}{2} \int \mathrm{d}x \, f G_0 \, \langle \pi^* \rangle, \tag{4.8}$$

from the symmetry of G_0, where $\langle \pi^* \rangle$ now denotes

$$\langle \pi^* \rangle = \sum_{r=1}^{n} p_r \pi_r(x). \tag{4.9}$$

To solve (4.7), it is convenient to eliminate $G_0 f$ in favour of $\langle u^* \rangle$, by employing the relation

$$\langle u^* \rangle = G_0(f - s\langle \pi^* \rangle). \tag{4.10}$$

Thus,

$$p_r \pi_r / (\rho_r - \rho_0) + s^2 \sum_{s=1}^{n} G_0 (p_{rs} - p_r p_s) \pi_s = s p_r \langle u^* \rangle. \tag{4.11}$$

Suppose that solving these equations yields

$$\langle \pi^* \rangle = s \Pi \langle u^* \rangle \tag{4.12}$$

for some operator Π. Then, from eqns (4.10) and (4.12),

$$\langle u^* \rangle = \tilde{G} f, \tag{4.13}$$

where

$$\bar{G} = (I + s^2 G_0 \Pi)^{-1} G_0. \tag{4.14}$$

If, correspondingly,

$$\langle u \rangle = \hat{G} f, \tag{4.15}$$

it follows from (3.18), (4.10) and (4.13) that

$$\frac{1}{2} \int \mathrm{d}x\, f \hat{G} f \underset{(\leq)}{\geq} \frac{1}{2} \int \mathrm{d}x\, f \bar{G} f, \tag{4.16}$$

when $\rho_0 \geq (\leq)\ \rho_r$ for all r.

The result (4.16) can be given symbolically in the shorter form

$$\hat{G} \underset{(\leq)}{\geq} \bar{G}, \tag{4.17}$$

the order relations implying (4.16). Correspondingly,

$$\hat{G}^{-1} \underset{(\geq)}{\leq} \bar{G}^{-1}. \tag{4.18}$$

But by definition of $\hat{G}$, $u = \hat{G} f$ implies

$$\operatorname{div}(Le) + f = s^2 \hat{\rho} u.$$

Therefore,

$$\hat{G}^{-1} u = s^2 \hat{\rho} u - \operatorname{div}(Le). \tag{4.19}$$

Also, inverting eqn (4.14),

$$\bar{G}^{-1} = G_0^{-1} + s^2 \Pi$$

and so, since G_0 is the Green's function for the comparison body,

$$\bar{G}^{-1} u = s^2 \bar{\rho} u - \operatorname{div}(Le), \tag{4.20}$$

where

$$\bar{\rho} = \rho_0 + \Pi. \tag{4.21}$$

Equation (4.20) demonstrates that $\bar{G}$ is the Green's function for a body whose "mass density operator" is $\bar{\rho}$. It may be noted, too, that (4.21) is also the estimate for $\hat{\rho}$ that would follow from (2.8) with $\langle \pi \rangle$ approximated by $\langle \pi^* \rangle$.

In view of (4.19) and (4.20), the result (4.18) can be given in the equivalent form

$$\hat{\rho} \underset{(\geq)}{\leq} \bar{\rho}, \tag{4.22}$$

whenever $\rho_r \leq (\geq)\ \rho_0$ for all r. It is emphasised again that the inequality symbols in (4.22) imply inequalities between quadratic forms, of the type that appear in (4.16). The development given does not rely on statistical uniformity. However, when the body has this property, equations (4.11) are insensitive to translations and the kernel of the operator Π, and hence $\bar{\rho}$, depends on $x - x'$ only. It is also an even function of $x - x'$, because the functions p_{rs} are even. The exact operator $\hat{\rho}$ shares these properties: this could be recognised by considering its representation as a series, as in Section 2, for example. Thus, both $\hat{\rho}$ and $\bar{\rho}$ are convolution operators with respect to x. Use of Parseval's theorem therefore gives

$$\int dx\, u(x)\, (\hat{\rho}u)\, (x) = \int dk\, \bar{u}(k)\hat{\rho}(k)u(k), \tag{4.23}$$

where $u(k)$ is written for the Fourier transform of $u(x)$ and $\bar{u}(k)$ is its complex conjugate, also equal to $u(-k)$ since $u(x)$ is real. The kernel of $\hat{\rho}$ is real and even in x, so $\hat{\rho}(k)$ is real and even in k. The inequalities (4.22) may therefore be given in the form

$$\int dk\, \bar{u}(k)[\hat{\rho}(k) - \bar{\rho}(k)]u(k) \underset{(\geq)}{\leq} 0, \tag{4.24}$$

for any choice of $u(k)$ such that $\bar{u}(k) = u(-k)$, and it follows that

$$\bar{u}[\hat{\rho}(k) - \bar{\rho}(k)]u \underset{(\geq)}{\leq} 0 \tag{4.25}$$

when $\rho_r \leq (\geq)\ \rho_0$ for all r, for every u and k. Thus, (4.22) implies inequalities for the eigenvalues of $\hat{\rho}(k) - \bar{\rho}(k)$, and bounds on the eigenvalues of $\hat{\rho}(k)$ whenever $\hat{\rho}$ and $\bar{\rho}$ are coaxial as they will be, for example, if the composite is isotropic. It is, perhaps, advisable to reiterate that $\hat{\rho}$ has already been Laplace transformed and that the results of this section apply when s is real.

5. EXAMPLE: TWO-PHASE COMPOSITE

The solution of eqns (4.11) takes a particularly simple form for a two-phase composite. The relations

$$p_1 + p_2 = 1, \qquad p_{11} + p_{12} = p_1, \qquad p_{12} + p_{22} = p_2 \tag{5.1}$$

imply, in the case of a statistically uniform medium, that

$$p_{11} = p_1^2 + p_1p_2h, \qquad p_{12} = p_1p_2(1 - h), \qquad p_{22} = p_2^2 + p_1p_2h, \tag{5.2}$$

where the function h depends upon $x - x'$ and $h(0) = 1$. Also, $h \to 0$ as $|x - x'| \to \infty$ so long as the composite displays no long-range order. Direct calculation shows that, for the two-phase composite, the function h in (5.2) coincides with the h defined for a general composite by (2.10).

Substitution of (5.2) into (4.11) gives

$$\pi_r/(\rho_r - \rho_0) + s^2H(\pi_r - \langle\pi^*\rangle) = s\langle u^*\rangle, \qquad r = 1, 2, \tag{5.3}$$

where H is now the Laplace transform of the operator defined by (2.12). A formal solution to these equations is easy to obtain and yields

$$\Pi = \{\textstyle\sum p_r\,[I + s^2(\rho_r - \rho_0)H]^{-1}\}^{-1}\{\textstyle\sum p_s(\rho_s - \rho_0)[I + s^2(\rho_s - \rho_0)H]^{-1}\}, \tag{5.4}$$

for use in conjunction with (4.12). Since the operations in (5.4) are convolutions, Fourier transforming leaves (5.4) unchanged, except that now the operations are just matrix multiplications.

Bounds for $\hat{\rho}$ are obtained from (5.4) by choosing $\rho_0 = \rho_1$ or ρ_2. Assuming, to be definite, that $\rho_1 > \rho_2$, the choice $\rho_0 = \rho_2$ generates a lower bound. When $\rho_0 = \rho_2$, (5.4) simplifies and $\bar{\rho}$ can be given in the form

$$\bar{\rho} = \rho_2 + p_1(\rho_1 - \rho_2)[I + p_2 s^2(\rho_1 - \rho_2)H]^{-1}. \quad (5.5)$$

The corresponding upper bound, obtained by setting $\rho_0 = \rho_1$, is obtained by interchanging the subscripts 1 and 2 in (5.5).

So far, the bounds given from (5.5) apply to a composite whose tensor of moduli L may be anisotropic, and which may have any correlation function h. Explicit results are now given, for the case of an isotropic composite, so that L is isotropic and h is a function of $|x - x'|$ only. If L is characterized by Lamé moduli λ, μ, the comparison material has wave speeds α, β, where

$$\alpha^2 = (\lambda + 2\mu)/\rho_0, \qquad \beta^2 = \mu/\rho_0 \quad (5.6)$$

and the Fourier transformed Green's function $G_0(k, s)$ satisfies the equations

$$(\beta^2 |k|^2 + s^2)(G_0)_{ij} + (\alpha^2 - \beta^2)k_j k_p (G_0)_{ip} = \delta_{ij}/\rho_0. \quad (5.7)$$

Therefore, solving (5.7),

$$[G_0(k, s)]_{ij} = \frac{1}{\rho_0}\left\{\frac{\delta_{ij}}{\beta^2|k|^2 + s^2} + \frac{k_i k_j}{s^2}\left[\frac{\beta^2}{\beta^2|k|^2 + s^2} - \frac{\alpha^2}{\alpha^2|k|^2 + s^2}\right]\right\}. \quad (5.8)$$

Inverting (5.8) gives

$$[G_0(x, s)]_{ij} = \frac{1}{4\pi\rho_0}\left\{\delta_{ij}\frac{e^{-s|x|/\beta}}{\beta^2|x|} + \frac{1}{s^2}\frac{\partial^2}{\partial x_i \partial x_j}\left[\frac{e^{-s|x|/\alpha} - e^{-s|x|/\beta}}{|x|}\right]\right\}. \quad (5.9)$$

These results are, of course, well-known.

It is necessary now to multiply (5.9) by $h(x)$ and then transform back. For the purpose of illustration, this procedure is carried through for the particular correlation function

$$h(x) = e^{-|x|/a}. \quad (5.10)$$

Since the resulting $H(k, s)$ is an isotropic function, it can be represented in the form

$$[H(k, s)]_{ij} = \frac{k_i k_j}{|k|^2} H_{\rm I}(|k|) + \left(\delta_{ij} - \frac{k_i k_j}{|k|^2}\right) H_{\rm II}(|k|). \quad (5.11)$$

The advantage of this form is that H has eigenvalues $H_{\rm I}$, $H_{\rm II}$ and, if H is represented by

$$H = (H_{\rm I}, H_{\rm II}) \quad (5.12)$$

and K is an isotropic function, represented similarly, their product has representation

$$HK = (H_{\rm I}K_{\rm I}, H_{\rm II}K_{\rm II}). \quad (5.13)$$

Also, since the identity has representation (1, 1),

$$H^{-1} = (1/H_{\rm I}, 1/H_{\rm II}). \quad (5.14)$$

To find $H_{\rm I}$, $H_{\rm II}$, it may be noted that

$$[H(k, s)]_{jj} = H_{\rm I} + 2H_{\rm II} \quad (5.15)$$

and, when $k = (0, 0, |k|)$,

$$[H(k, s)]_{33} = H_{\rm I}. \quad (5.16)$$

Considering first H_{jj}, it follows from (5.9) that

$$[G_0(x, s)]_{jj} = \frac{1}{4\pi\rho_0}\left\{\frac{e^{-s|x|/\alpha}}{\alpha^2 |x|} + \frac{2e^{-s|x|/\beta}}{\beta^2 |x|}\right\}. \tag{5.17}$$

Multiplying by $e^{-|x|/a}$ and Fourier transforming then gives

$$H_I + 2H_{II} = \frac{1}{\rho_0}\left\{\frac{1}{\alpha^2[\,|k|^2 + (s/\alpha + 1/a)^2]} + \frac{2}{\beta^2[\,|k|^2 + (s/\beta + 1/a)^2]}\right\}. \tag{5.18}$$

Evaluating H_I using (5.16) requires rather more work. First,

$$\begin{aligned}[H(x, s)]_{33} = \frac{1}{4\pi\rho_0}\Bigg\{&\frac{(3\cos^2\theta - 1)}{s^2}\left[\left(\frac{1}{|x|^3} + \frac{s}{\alpha|x|^2}\right)e^{-(s/\alpha + 1/a)|x|}\right.\\ &\left.-\left(\frac{1}{|x|^3} + \frac{s}{\beta|x|^2}\right)e^{-(s/\beta + 1/a)|x|}\right]\\ &+\cos^2\theta\left[\frac{e^{-(s/\alpha + 1/a)|x|}}{\alpha^2|x|} - \frac{e^{-(s/\beta + 1/a)|x|}}{\beta^2|x|}\right]\\ &+\frac{e^{-(s/\beta + 1/a)|x|}}{\beta^2|x|}\Bigg\},\end{aligned} \tag{5.19}$$

where $\cos\theta = x_3/|x|$. Taking the Fourier transform of (5.19) involves $\exp(ik.x)$ and, when $k = (0, 0, |k|)$, this reduces to $\exp(i|k||x|\cos\theta)$. The relevant integrals are summarised in the Appendix; the final result is

$$H_I = \frac{1}{\rho_0}\left\{F(\alpha) - F(\beta) + \frac{1}{\beta^2|k|^2 + (s + \beta/a)^2}\right\}, \tag{5.20}$$

$$H_{II} = \frac{1}{2\rho_0}\left\{F(\beta) - F(\alpha) + \frac{1}{\alpha^2|k|^2 + (s + \alpha/a)^2} + \frac{1}{\beta^2|k|^2 + (s + \beta/a)^2}\right\}, \tag{5.21}$$

where

$$\begin{aligned}F(\gamma) = &\left\{\frac{(s/\gamma + 1/a)}{s^2|k|}\left[1 + \frac{(s/\gamma + 1/a)^2}{|k|^2}\right] - \frac{1}{\gamma s|k|}\left[1 + \frac{3(s/\gamma + 1/a)^2}{|k|^2}\right]\right.\\ &\left.+\frac{2(s/\gamma + 1/a)}{\gamma^2|k|^3}\right\}\tan^{-1}\left(\frac{|k|}{s/\gamma + 1/a}\right)\\ &-\frac{(s/\gamma + 1/a)^2}{s^2|k|^2} + \frac{3(s/\gamma + 1/a)}{\gamma s|k|^2} - \frac{1}{\gamma^2|k|^2}\left[1 + \frac{(s/\gamma + 1/a)^2}{|k|^2 + (s/\gamma + 1/a)^2}\right].\end{aligned} \tag{5.22}$$

Then, from eqn (5.5), when $\rho_0 = \rho_2$,

$$\bar{\rho} = \left\{\rho_2 + \frac{p_1(\rho_1 - \rho_2)}{1 + p_2 s^2(\rho_1 - \rho_2)H_I},\quad \rho_2 + \frac{p_1(\rho_1 - \rho_2)}{1 + p_2 s^2(\rho_1 - \rho_2)H_{II}}\right\} \tag{5.23}$$

and each component in (5.23) provides a lower bound for the corresponding component of $\hat{\rho}(k, s)$. As mentioned above, upper bounds are obtained simply by interchanging ρ_1 and ρ_2, and p_1 and p_2.

Bounds have been calculated, using a simple computer program that evaluates (5.23), for a variety of choices for the parameters ρ_1/ρ_2, p_1, $|k|\,a$ and as/β_2, where $\beta_2 = (\mu/\rho_2)^{1/2}$ is the faster of the two shear wave speeds. Poisson's ratio was taken as

Table 1. Bounds for the eigenvalues of $\hat{\rho}$ for the case $\rho_1/\rho_2 = 2$, $p_1 = 0.3$, when $as/\beta_2 = 2$.

$\log_2(\|k\|a)$	$\tilde{\rho}_I/\rho_2$		$\tilde{\rho}_{II}/\rho_2$	
	lower	upper	lower	upper
-5	1.235	1.244	1.235	1.244
-4	1.235	1.244	1.235	1.244
-3	1.236	1.244	1.235	1.244
-2	1.236	1.245	1.236	1.244
-1	1.238	1.246	1.237	1.245
0	1.245	1.252	1.240	1.248
1	1.261	1.265	1.251	1.256
2	1.283	1.284	1.272	1.273
3	1.295	1.295	1.289	1.290
4	1.299	1.299	1.297	1.297
5	1.300	1.300	1.299	1.299

Table 2. Bounds for the eigenvalues of $\hat{\rho}$ for the case $\rho_1/\rho_2 = 4$, $p_1 = 0.5$, $as/\beta_2 = 1$.

$\log_2(\|k\|a)$	$\tilde{\rho}_I/\rho_2$		$\tilde{\rho}_{II}/\rho_2$	
	lower	upper	lower	upper
-5	2.139	2.242	2.139	2.242
-4	2.139	2.242	2.139	2.242
-3	2.141	2.242	2.140	2.242
-2	2.146	2.245	2.143	2.243
-1	2.164	2.256	2.154	2.249
0	2.223	2.292	2.192	2.269
1	2.340	2.370	2.286	2.325
2	2.444	2.479	2.403	2.411
3	2.486	2.486	2.469	2.470
4	2.497	2.497	2.492	2.492
5	2.499	2.499	2.498	2.498

Table 3. Bounds for the eigenvalues of $\hat{\rho}$ for the case $\rho_1/\rho_2 = 10$, $p_1 = 0.3$, when $as/\beta_2 = 1$.

$\log_2(\|k\|a)$	$\tilde{\rho}_I/\rho_2$		$\tilde{\rho}_{II}/\rho_2$	
	lower	upper	lower	upper
-5	2.158	2.662	2.158	2.662
-4	2.159	2.662	2.159	2.662
-3	2.162	2.664	2.160	2.663
-2	2.174	2.671	2.167	2.666
-1	2.221	2.697	2.195	2.678
0	2.384	2.791	2.295	2.722
1	2.797	3.038	2.590	2.868
2	3.319	3.386	3.094	3.181
3	3.594	3.602	3.482	3.493
4	3.675	3.675	3.638	3.639
5	3.694	3.694	3.684	3.684

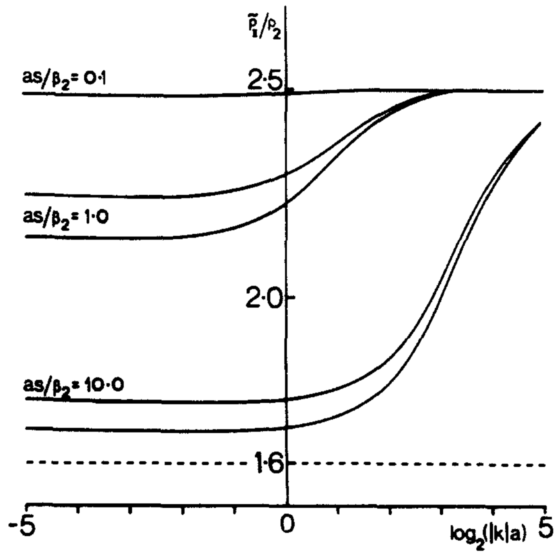

Fig. 1. Bounds for $\hat{\rho}_I/\rho_2$, plotted against $|k|a$, for the three values $as/\beta_2 = 0.1, 1, 10$. The composite has $\rho_1/\rho_2 = 4$, $p_1 = 0.5$, and Poisson's ratio is 0.25.

0.25, so that $\lambda = \mu$. Sample results are presented in Tables 1–3 and in Fig. 1. Table 1 shows lower and upper bounds for $\hat{\rho}_I(k, s)$ and $\hat{\rho}_{II}(k, s)$, for a medium with density ratio $\rho_1/\rho_2 = 2$ and $p_1 = 0.3$, when $as/\beta_2 = 2$, for a range of values of $|k|a$. There is less than one percent difference between the bounds, so that, if $\hat{\rho}$ were estimated as lying mid-way between the bounds, the error would be at most 0.5 percent. The difference between $\hat{\rho}_I$ and $\hat{\rho}_{II}$ is small, so that the tensor $\hat{\rho}$ is almost a scalar (that is, a scalar times the Kronecker delta). It may be noted, however, that for values of $|k|a$ in the range 2 to 8, the lower bound for $\hat{\rho}_I$ is greater than the upper bound for $\hat{\rho}_{II}$, so that $\hat{\rho}$ is definitely not exactly a scalar operator. The table also shows clearly that $\hat{\rho}$ varies with k so that it is a non-local operator. Tables 2 and 3 show similar results, for larger density ratios. In Table 2, $\rho_1/\rho_2 = 4$, $p_1 = 0.5$ and $as/\beta_2 = 1$. The bounds differ by up to five percent. Since this difference is visible on a diagram, this density ratio and volume fraction were selected for the graphical presentation of Fig. 1, which shows lower and upper bounds for $\hat{\rho}_I(k, s)$, plotted against $|k|a$, for the three choices $as/\beta_2 = 0.1, 1, 10$. Thus, the middle pair of curves corresponds exactly to Table 2. The figure displays trends that can also be confirmed analytically. For any fixed value of k, both the lower- and upper-bound estimates for $\hat{\rho}_I$, $\hat{\rho}_{II}$ (and therefore also $\hat{\rho}$ itself) tend to $\langle\rho\rangle$ (that is, just the mean density) as as/β_2 tends to zero. Also, for any fixed value of s, the same limit is approached as $|k|a$ tends to infinity. Finally, if $|k|a \to 0$ and $as/\beta_2 \to \infty$, $\hat{\rho}$ tends to the inverse "law of mixtures" value,

$$\hat{\rho} \to 1/\langle 1/\rho\rangle.$$

Furthermore, the eigenvalues of $\hat{\rho}(k, s)$ appear always to lie between these limits which, in analogy with terminology for elastic constants, might be termed Voigt ($\langle\rho\rangle$) and Reuss ($1/\langle 1/\rho\rangle$) estimates. For the parameter values used in Fig. 1, $\langle\rho\rangle = 2.5\rho_2$ and $1/\langle 1/\rho\rangle = 1.6\rho_2$. Table 3 shows what happens when the density ratio is increased to 10. The bounds become more widely separated (differing by up to 25%) but the difference between the Voigt and Reuss estimates also increases, so that $\hat{\rho}$ has more opportunity to vary. The bounds still give a good indication of this variation but higher order statistics would be needed to pin it down more precisely.

Finally, it is remarked that the expressions given for $\hat{\rho}(k, s)$ would remain valid if k and s were taken complex and could be used to provide approximate dispersion relations for mean plane waves. However, they would lose the precise status that they have as bounds when k and s are real.

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APPENDIX—SOME FOURIER TRANSFORMS

Here, the Fourier transforms of individual terms that appear in eqn (5.19) are recorded.

First, the term

$$\frac{(3\cos^2\theta - 1)}{4\pi\rho_0 s^2 |x|^3}[e^{-(s/\alpha + 1/a)|x|} - e^{-(s/\beta + 1/a)|x|}]$$

can be transformed, when $k = (0, 0, |k|)$, by employing polar coordinates $(|x|, \theta, \phi)$, to give

$$I_1 = \frac{1}{2\rho_0 s^2}\int_0^\infty \frac{d|x|}{|x|}\int_{-1}^{1} du(3u^2 - 1)e^{i|k||x|u}[e^{-(s/\alpha + 1/a)|x|} - e^{-(s/\beta + 1/a)|x|}], \tag{A.1}$$

having set $u = \cos\theta$ and performed the trivial integration with respect to ϕ. Integration by parts with respect to u gives

$$I_1 = \frac{i|k|}{2\rho_0 s^2}\int_0^\infty d|x|\int_{-1}^{1} du\,(u - u^3)e^{i|k||x|u}[e^{-(s/\alpha + 1/a)|x|} - e^{-(s/\beta + 1/a)|x|}]. \tag{A.2}$$

The integral with respect to $|x|$ is now elementary and so, after that, is the integral with respect to u that remains. The result is

$$I_1 = F_1(\alpha) - F_1(\beta), \tag{A.3}$$

where

$$F_1(\gamma) = \frac{1}{\rho_0 s^2}\left\{\frac{(s/\gamma + 1/a)}{|k|}\left(1 + \frac{(s/\gamma + 1/a)^2}{|k|^2}\right)\tan^{-1}\left(\frac{|k|}{s/\gamma + 1/a}\right) - \frac{(s/\gamma + 1/a)^2}{|k|^2}\right\}. \tag{A.4}$$

The term

$$\frac{(3\cos^2\theta - 1)}{4\pi\rho_0 s|x|^2}\left[\frac{e^{-(s/\alpha + 1/a)|x|}}{\alpha} - \frac{e^{-(s/\beta + 1/a)|x|}}{\beta}\right]$$

can be treated similarly. Its transform can be found from

$$F_2(\gamma) = \frac{1}{2\rho_0\gamma s}\int_0^\infty d|x|\int_{-1}^{1} du\,(3u^2 - 1)\exp\{[i|k|u - (s/\gamma + 1/a)]|x|\} \tag{A.5}$$

which, upon performing the elementary integrations, reduces to

$$F_2(\gamma) = \frac{1}{\rho_0\gamma s|k|}\left\{-\left(1 + \frac{3(s/\gamma + 1/a)^2}{|k|^2}\right)\tan^{-1}\left(\frac{|k|}{s/\gamma + 1/a}\right) + \frac{3(s/\gamma + 1/a)}{|k|}\right\}. \tag{A.6}$$

The term containing the factor $\cos^2\theta$ in (5.19) follows from

$$F_3(\gamma) = \frac{1}{2\rho_0\gamma^2}\int_0^\infty |x|\,d|x|\int_{-1}^{1} u^2 du\exp\{[i|k|u - (s/\gamma + 1/a)]|x|\}, \tag{A.7}$$

which reduces to

$$F_3(\gamma) = \frac{1}{\rho_0\gamma^2|k|^2}\left\{\frac{2(s/\gamma + 1/a)}{|k|}\tan^{-1}\left(\frac{|k|}{s/\gamma + 1/a}\right) - \left[1 + \frac{(s/\gamma + 1/a)^2}{|k|^2 + (s/\gamma + 1/a)^2}\right]\right\}. \tag{A.8}$$

The function $F(\gamma)$ in eqn (5.22) is given by

$$F(\gamma) = F_1(\gamma) + F_2(\gamma) + F_3(\gamma). \tag{A.9}$$